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Nonnegative Matrices which are Equal to Their Generalized Inverse

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ABSTRACT

Nonnegative matrices which are equal to their Moore-Penrose generalized inverse are characterized.

1. INTRODUCTION

In a recent paper, [4], Harary and Minc characterized the nonnegative matrices which are self-inverse.

In this note we study nonnegative matrices A which are equal to their Moore-Penrose generalized inverse, A^+, [6], defined by

$$A^+y = x \quad \text{if} \quad Ax = y,\, x \in R(A^T)$$

$$A^+y = 0 \quad \text{if} \quad y \in N(A^T),$$

or, equivalently, as the unique solution of the matrix equations

$$AXA = A \tag{1}$$

$$XAX = X \tag{2}$$

$$AX = (AX)^T \tag{3}$$

$$XA = (XA)^T. \tag{4}$$

The matrices we discuss are, of course, square. If, in addition, they are also nonsingular, then the generalized inverse is the inverse and the problem in hand is the one of Harary and Minc.

2. THE CHARACTERIZATION

THEOREM *Let A be a nonnegative matrix. Then $A = A^+$ if and only if A is square and there exists a permutation matrix P such that PAP^T is a direct sum of square matrices of the following (not necessarily all) three types:*

(i) xx^T, *where x is a positive vector such that $x^Tx = 1$.*

(ii)
$$\begin{pmatrix} 0 & xy^T \\ dyx^T & 0 \end{pmatrix},$$

where x and y are positive vectors (not necessarily of the same order), $d > 0$, $dx^Txy^Ty = 1$ and the 0's stand for square matrices of the appropriate sizes.

(iii) *a zero matrix.*

Proof. If A is $m \times n$ then A^+ is $n \times m$. Thus if $A = A^+$ it has to be square. From (1) and (3) (or (2) and (4)) it follows that $A = A^+$ if and only if

$$A^3 = A \tag{5}$$

and

$$A^2 \text{ is symmetric.} \tag{6}$$

A necessary and sufficient condition due to Plemmons and Cline [7, 1], for a nonnegative matrix A to have a nonnegative generalized inverse is

$$A = DA^T \tag{7}$$

for some diagonal matrix $D = \{\operatorname{diag} d_i\}$, $d_i > 0$. Thus (7) is a necessary condition for $A = A^+ \geqslant 0$.

Since A is nonnegative, we can reduce it by a suitable permutation to triangular block form

$$PAP^T = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix},$$

where the diagonal blocks are square irreducible (indecomposable [8]) or zero matrices, e.g. [5, p. 75].

The cogredient permutation is an isomorphism for matrix multiplication and symmetry is invariant by it, thus $A=A^+$ if and only if $(PAP^T)^+=PAP^T$, (P being a permutation matrix), and PAP^T has to satisfy (7). Thus, the blocks A_{ij}, $i>j$, are zero matrices and we conclude that the normal form of A [5, p. 75] is a diagonal block matrix:

$$PAP^T=\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}, \tag{8}$$

where the matrices A_i are square irreducible or zero matrices.

We now show that the matrices A_i are of the types given in the theorem.

From (5) it follows that the minimal polynomials of the matrices A_i (and of A) divide $\lambda^3-\lambda$. Thus A and the matrices A_i are similar to diagonal matrices and their eigenvalues can be 0, 1, or -1.

The matrix A_i is thus either a zero matrix (type iii) or by the Frobenius Theorem, e.g. [5, p. 53] has 1 as a simple majoring eigenvalue. By the Frobenius Theorem this is possible in two cases: i) A_i is primitive [5, p. 80], ii) it has 2 as index of imprimitivity.

Suppose A_i is primitive. Then 1 is a simple eigenvalue and all other eigenvalues (if any) are zeros. Thus trace $A_i=1$ and since A_i is diagonalizable, rank $A_i=1$ which means that

$$A_i=xy^T, \tag{9}$$

where x and y are positive vectors of the same size. (If x or y had a zero coordinate, A_i would have a zero row (or column) contradicting its irreducibility.) Now by (7),

$$A_i=DA_i^T$$

$$D=\{\operatorname{diag} d_k\}, d_k>0$$

but $A_i>0 \Rightarrow \operatorname{diag} A_i>0 \Rightarrow D=I \Rightarrow A_i=A_i^T$.

The symmetry of A_i follows also from its being idempotent, which follows from

$$A_i=S^{-1}\operatorname{diag}\{1,0,\ldots,0\}S,$$

and from (6), and it implies that y in (9) may be taken as x. The condition that trace $A_i=1$ means that $x^Tx=1$ and thus A_i is of type i).

Suppose now that the index of imprimitivity of A_i is 2. Then the eigenvalues of A_i are 1, -1 and possibly zeros, and, again by the Frobenius Theorem, the matrix A_i, is cogredient to

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \tag{10}$$

where the zero blocks are square matrices (of possibly different orders).

By (7)

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 0 & C^T \\ B^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_1C^T \\ D_2B^T & 0 \end{pmatrix},$$

where $D_1 = \{\operatorname{diag} e_\ell\}$, $e_\ell > 0$, $D_2 = \{\operatorname{diag} d_m\}$, $d_m > 0$ are diagonal matrices of the appropriate sizes.

Thus

$$B = D_1 C^T \tag{11}$$

$$C = D_2 B^T. \tag{12}$$

Since A_i is diagonalizable, its rank is 2 and rank $B =$ rank $C = 1$. Thus B and C are positive matrices (otherwise A_i would have a zero row or column contradicting its irreducibility), and this together with (11) and (12) implies that D_1 and D_2 are scalar matrices: $e_\ell = 1/d$, $d_m = d$.

Thus (10) reduces to the form

$$\begin{pmatrix} 0 & xy^T \\ dyx^T & 0 \end{pmatrix},$$

where x and y are positive vectors of the appropriate sizes.

In order that the nonzero eigenvalues of A_i be 1 and -1, the sum of its principal minors of order 2 has to be 1, but this sum is $-dx^Txy^Ty$, which shows that A_i is of type (ii). This completes the "only if" part of the theorem.

To prove the "if" part, recall the remark that $(PAP^T)^+ = PAP^T$ if and only if $A^+ = A$, and check (5) and (6) for each of the three types of the theorem. They follow from the idempotency and symmetry of the matrices of types (i) and (iii) and are easily checked for matrices of type (ii). ■

3. REMARKS

1. $A = A^+ \geqslant 0$ is idempotent if and only if it is symmetric idempotent and if and only if its normal form (8) does not contain matrices of type (ii).

This characterization of symmetric nonnegative idempotent matrices is compatible with the characterization of nonnegative idempotents given by Flor [3].

2. $A=A^{+}\geqslant 0$ is nonsingular, that is A is self-inverse, if and only if the matrices in the normal form (8) are matrices of type (i) of order 1 or matrices of type (ii) of order 2. Observing that these matrices are either (1) or

$$\begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix},$$

one gets the characterization given by Harary and Minc [4].

3. $A=A^{+}\geqslant 0$ has the property that A and A^{T} have a positive eigenvector corresponding to the eigenvalue 1, if and only if there are no matrices of type (iii) in the normal form (8).

4. $A=A^{+}>0$ if and only if A itself is a block of type (i).

5. Let λ be any subset of $\{1,2,3,4\}$ which contains 1. A λ-inverse of A is a solution of equations (t), $t\in\lambda$. For example, A^{+} is the $\{1,2,3,4\}$-inverse of A.

A characterization of all nonnegative matrices having a nonnegative λ-inverse, is given in [2], for all possible λ. On these lines, one may wish to look at nonnegative matrices which are equal to a λ-inverse of themselves. It is clear that if $3\in\lambda$ or $4\in\lambda$, then $A=A^{+}$ and the problem is the one discussed above. It gets more complicated if $\lambda=\{1\}$ or $\lambda=\{1,2\}$. In this case, the normal form (7) of A needs not be quasi-diagonal. A simple example is furnished by

$$A=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

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